NEAR-EXTREMIZERS OF YOUNG'S INEQUALITY FOR DISCRETE GROUPS

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Abstract. Those functions which nearly extremize Young's convolution inequality are characterized for discrete groups which have no nontrivial finite subgroups. Near-extremizers of the Hausdorff-Young inequality are characterized for $\mathbb{Z}^d$.

1. Introduction

Let G be any discrete group. We denote the product of $x, y \in G$ by $x + y$, even though G is not assumed to be Abelian. Equip G with counting measure, so that $\|f\|_{L^p(G)} = \big(\sum_{x\in G} |f(x)|^p\big)^{1/p}$ for any $p \in [1,\infty)$, while $\|f\|_\infty = \sup_{x\in G} |f(x)|$.

Write
$$\langle f_1 * f_2, f_3\rangle = \sum_{x,y\in G} f_1(x) f_2(y) f_3(x+y).$$
Young's convolution inequality states that for any discrete group,
$$\Big|\langle f_1 * f_2, f_3\rangle\Big| \le \prod_{j=1}^{3} \|f_j\|_{p_j} \tag{1.1}$$
for complex-valued functions whenever each $p_j \in [1,\infty]$ and $\sum_j p_j^{-1} = 2$. Equivalently, $\|f_1 * f_2\|_q \le \|f_1\|_{p_1}\|f_2\|_{p_2}$ for $q^{-1} = p_1^{-1} + p_2^{-1} - 1$ provided $q \in [1,\infty]$.

The optimal constants in these inequalities equal 1, since equality is attained if each f_j is supported on a single point z_j, provided that $z_3 = z_1 + z_2$. If $G = \mathbb{Z}^d$ for some $d \ge 1$, then it is an elementary fact that so long as all p_j belong to $(1,\infty)$, equality is attained only if the support of each f_j is a singleton, with the relation $z_3 = z_1 + z_2$ between their respective supports $\{z_j\}$. This last assertion does not hold for general groups, for if H is a finite subgroup of G and each function f_j is the indicator function of H, then equality holds in (1.1).

Another relevant example is as follows. Let $G = \mathbb{Z}$. Let N be a large positive integer. Define $f_j^{(N)}$ to be the indicator function of the interval $[-N, N]$ for $j = 1, 2, 3$. Then $\prod_j \|f_j^{(N)}\|_{p_j} = (2N+1)^2 = 4N^2 + O(N)$ provided that $\sum_j p_j^{-1} = 2$, while $\langle f_1^{(N)} * f_2^{(N)}, f_3^{(N)}\rangle = 3N^2 + O(N)$. Thus as $N \to \infty$,
$$|\langle f_1^{(N)} * f_2^{(N)}, f_3^{(N)}\rangle| \ge \Big(\frac{3}{4} - O(N^{-1})\Big) \prod_{j=1}^{3} \|f_j^{(N)}\|_{p_j}.$$

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Yet any subsequence of $f_1^{(N)}/\|f_1^{(N)}\|_{p_1}$, even modulo translations, converges weakly to 0. The main result of this paper implies that such a phenomenon cannot arise if $\frac{3}{4}$ is replaced by a constant sufficiently close to 1.

Definition 1.1. Let exponents $p_j \in (1,\infty)$ satisfy $\sum_{j=1}^{3} p_j^{-1} = 2$. A triple of nonzero functions $(f_1, f_2, f_3) \in \times_{j=1}^{3} L^{p_j}$ is a $(1-\delta)$–near extremizer for Young's inequality if

$$|\langle f_1 * f_2, f_3\rangle| \ge (1-\delta)\prod_{j=1}^{3}\|f_j\|_{p_j}. \tag{1.2}$$

Definition 1.2. A group is said to be torsion-free if it has no finite subgroups except the trivial subgroup of cardinality equal to one.

Theorem 1.1. *For any $d \ge 1$ and any ordered triple of exponents $p_j \in (1,\infty)$ satisfying $\sum_j p_j^{-1} = 2$, there exists a function $\delta : (0,1] \to (0,\infty)$ satisfying $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ with the following property. Let G be any torsion-free discrete group. For any $\varepsilon \in (0,1]$, if a triple (f_1, f_2, f_3) is a $(1-\delta(\varepsilon))$–near extremizer for Young's inequality for G, then there exist points $z_j \in G$ and scalars $c_j \in \mathbb{C}$ such that for each index $j \in \{1,2,3\}$,*

$$\|f_j - c_j \mathbf{1}_{\{z_j\}}\|_{p_j} < \varepsilon \|f_j\|_{p_j}. \tag{1.3}$$

Moreover $z_3 = z_1 + z_2$.

Equivalently, $\|f_j\|_\infty \ge (1-\varepsilon)\|f_j\|_{p_j}$, for a different but comparable value of ε. A more quantitative formulation is as follows.

Theorem 1.2. *Let $p_1, p_2, q \in (1,\infty)$ satisfy $q^{-1} = p_1^{-1} + p_2^{-1} - 1$. There exist a continous nondecreasing function $\Lambda : (0,1] \to (0,1]$, an exponent $\gamma > 0$, and a constant $c > 0$ satisfying*

$$\Lambda(t) \le 1 - c(1-t)^\gamma \ \text{as } t \to 1^-$$

such that for any torsion-free discrete group G and for any functions $f_i \in L^{p_i}(G)$,

$$\|f_1 * f_2\|_q \le \|f_1\|_{p_1}\|f_2\|_{p_2} \cdot \Lambda\Big(\frac{\|f_1\|_\infty}{\|f_1\|_{p_1}}\Big) \cdot \Lambda\Big(\frac{\|f_2\|_\infty}{\|f_2\|_{p_2}}\Big). \tag{1.4}$$

The analogue of Theorem 1.1 for $\mathbb{R}^d$, has been proved in [1]. The simple method of proof developed here cannot apply to $\mathbb{R}^d$. Nonetheless, the questions for $\mathbb{R}$ and for $\mathbb{Z}$ are connected; the following reasoning shows that if Theorem 1.1 were not valid for $\mathbb{Z}$, then its analogue for $\mathbb{R}$ would also necessarily fail. Indeed, fix an extremizer $F \in L^p(\mathbb{R})$ for Young's inequality for $\mathbb{R}$. If a sequence of functions $f_\nu \in L^p(\mathbb{Z})$ were to furnish a counterexample to Theorem 1.1 for $\mathbb{Z}$, then the sequence of functions $g_\nu(x) = \sum_{n\in\mathbb{Z}} f_\nu(n)\nu^{1/p}F(\nu x)$ would provide a counterexample for $\mathbb{R}$.

The Hausdorff-Young inequality for $\mathbb{Z}^d$ states that

$$\|\widehat{f}\|_{L^{p'}(\mathbb{T}^d)} \le \|f\|_{\ell^p(\mathbb{Z}^d)} \tag{1.5}$$

for all $1 \le p \le 2$, where $p' = p/(p-1)$, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, equipped with the measure which pulls back to Lebesgue measure on $\mathbb{R}^d$ under the natural projection of $\mathbb{R}^d$ onto $\mathbb{T}^d$, $\widehat{f}(\xi) = \sum_{x\in\mathbb{Z}^d} e^{-2\pi i\xi\cdot x} f(x)$, and $\|g\|_{L^q(\mathbb{T}^d)} = \big((2\pi)^{-d}\int_{\mathbb{T}^d}|g(y)|^q\,dy\big)^{1/q}$.

Our results for Young's inequality have the following implication for the Hausdorff-Young inequality.

Theorem 1.3. *Let $p \in (1,2)$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any dimension d and any $(1-\delta)$–near extremizer $f \in L^p(\mathbb{Z}^d)$ for the Hausdorff-Young inequality, there exists $z \in \mathbb{Z}^d$ such that $\|f\|_{L^p(\mathbb{Z}^d \setminus \{z\})} < \varepsilon \|f\|_p$. More precisely, there exist a continuous nondecreasing function $\Lambda : (0,1] \to (0,1]$, an exponent $\gamma \in (0,\infty)$, and a constant $c > 0$ satisfying*

$$\Lambda(t) \le 1 - c(1-t)^\gamma \ \text{as } t \to 1^-$$

such that for any $d \ge 1$ and for any function $f \in \ell^p(\mathbb{Z}^d)$,

$$\|\widehat{f}\|_{p'} \le \|f\|_p \cdot \Lambda\Big(\frac{\|f\|_\infty}{\|f\|_p}\Big). \tag{1.6}$$

2. Lorentz space bound, and a consequence

We will work with triples of exponents $\mathbf{p} = (p_j : 1 \le j \le 3)$ which satisfy the two hypotheses

$$\sum_j p_j^{-1} = 2 \tag{2.1}$$

$$p_j \in (1,\infty) \ \text{ for all } j \in \{1,2,3\}\,. \tag{2.2}$$

Let G be any discrete group. Let $(f_j)_{1 \le j \le 3}$ be a triple of functions with $f_j \in L^{p_j}$. If $|\langle f_1 * f_2, f_3 \rangle| \ge (1-\delta)\prod_{j=1}^3 \|f_j\|_{p_j}$, then the same holds with each f_j replaced by $|f_j|$. If $|f_j|$ satisfies the desired conclusion, then so does f_j. Therefore it suffices to work only with nonnegative functions in the sequel.

Recall the Lorentz spaces $L^{p,r}$, where $p \in (1,\infty)$ and $r \in [1,\infty]$ [4]. If $f = \sum_{k\in\mathbb{Z}} 2^k F_k$ where $\mathbf{1}_{E_k} \le F_k \le 2\mathbf{1}_{E_k}$ and the sets E_k are pairwise disjoint, then the Lorentz norm $\|f\|_{L^{p,r}}$ satisfies

$$\|f\|_{L^{p,r}} \asymp \Big(\sum_{k=-\infty}^{\infty} (2^k |E_k|^{1/p})^r \Big)^{1/r}$$

in the sense that one of these quantities is finite if and only if the other is finite, and each is majorized by a constant multiple of the other, where these constants depend only on the indices p, r. In particular, if $p < r < \infty$ then there exist $\eta \in (0,1)$ and $C < \infty$ such that for any $f \in L^{p,r}$,

$$\|f\|_{p,r} \le C \|f\|_p^{1-\eta} \big(\sup_k 2^k |E_k|^{1/p} \big)^\eta.$$

By real interpolation [4], for any $\mathbf{p}$ satisfying (2.1) and (2.2), there exist exponents $r_j > p_j$ such that

$$\langle f_1 * f_2, f_3 \rangle \le C \prod_{j=1}^3 \|f_j\|_{L^{p_j, r_j}} \tag{2.3}$$

where $C < \infty$ depends only on $\mathbf{p}$.

By Hölder's inequality, (2.3) implies that there exists $\eta = \eta(p_1, p_2, p_3) > 0$ such that for all $\{f_j\}$,

$$\langle f_1 * f_2, f_3\rangle \le C \prod_{j=1}^{3} \|f_j\|_{L^{p_j}}^{1-\eta} \prod_{j=1}^{3} \sup_{k_j\in\mathbb{Z}} (2^{k_j}|E_{j,k_j}|^{1/p_j})^{\eta}. \tag{2.4}$$

Therefore

Lemma 2.1. *Let* $\mathbf{p}$ *satisfy* (2.1) *and* (2.2). *For any* $\delta > 0$ *there exists* $\rho > 0$ *such that for any nonnegative functions* f_j *satisfying* $\langle f_1 * f_2, f_3\rangle \ge \delta \prod_{j=1}^{3} \|f_j\|_{p_j}$, *for each* $j \in \{1, 2, 3\}$ *there exists* $\kappa_j \in \mathbb{Z}$ *such that*

$$2^{\kappa_j}|E_{j,\kappa_j}|^{1/p_j} \ge \rho\|f_j\|_{p_j}. \tag{2.5}$$

3. A consequence of a proof of Young's inequality

One proof of Young's inequality uses Hölder's inequality to reduce matters to the simpler inequality $\|f * g\|_p \le \|f\|_1\|g\|_p$. In this section we review this proof in order to extract additional information from it.

Lemma 3.1. *Let* G *be any discrete group. For any* $\mathbf{p}$ *satisfying hypotheses* (2.1) *and* (2.2), *there exist exponents* $s_1, s_2 \in (1,\infty)$ *and a constant* $C < \infty$ *such that for any* $\delta \in [0, 1]$, *if* $\mathbf{f} = (f_j : 1 \le j \le 3)$ *is any* $(1-\delta)$*–near extremizer for Young's inequality for* G *with exponents* $\mathbf{p}$, *then* $\mathbf{g} = (|f_1|^{p_1/s_1}, |f_2|^{p_2/s_2}, |f_3|^{p_3})$ *is a* $(1 - C\delta)$*–near extremizer for Young's inequality with exponents* $(s_1, s_2, 1)$.

Moreover, the same conclusion holds for any permutation of the three indices.

Proof. Without loss of generality, we may assume that each function f_j is nonnegative. Let $q = p_3'$ be the exponent conjugate to p_3, set $\theta = p_1/q \in (0, 1)$ and $\phi = p_2/q \in (0, 1)$, and write the cubic form as

$$\begin{aligned}\langle f_1 * f_2, f_3\rangle &= \sum_{x,y} f_1(x)f_2(y)f_3(x + y)\\ &= \sum_{x,y} \Big(f_1(x)^{1-\theta} f_2(y)^{1-\phi} f_3(x + y)\Big) \cdot \Big(f_1(x)^{\theta} f_2(y)^{\phi}\Big)\\ &\le \Big(\sum_{x,y} f_1(x)^{(1-\theta)p_3} f_2(y)^{(1-\phi)p_3} f_3(x + y)^{p_3}\Big)^{1/p_3} \cdot \Big(\sum_{x,y} f_1(x)^{p_1} f_2(y)^{p_2}\Big)^{1/q}\\ &= \Big(\sum_{x,y} f_1(x)^{r_1} f_2(y)^{r_2} f_3(x + y)^{p_3}\Big)^{1/p_3} \|f_1\|_{p_1}^{p_1/q}\|f_2\|_{p_2}^{p_2/q}\end{aligned}$$

where $r_1 = (q - p_1)p_3/q$ and $r_2 = (q - p_2)p_3/q$. Define $s_j = p_j/r_j$ for $j = 1, 2$, and $s_3 = 1$. Then $f_3^{p_3} \in L^1$, while $g_j = f_j^{r_j} \in L^{s_j}$. Moreover

$$\begin{aligned}\frac{1}{s_1} + \frac{1}{s_2} = \frac{r_1}{p_1} + \frac{r_2}{p_2} &= \frac{(q - p_1)p_3}{qp_1} + \frac{(q - p_2)p_3}{qp_2}\\ &= p_3\Big(\frac{1}{p_1} - \frac{1}{q} + \frac{1}{p_2} - \frac{1}{q}\Big) = p_3\Big(\frac{1}{p_1} + \frac{1}{p_2} - 2 + \frac{2}{p_3}\Big) = 1\end{aligned}$$

since $\sum_{j=1}^3 \frac{1}{p_j} = 2$. Therefore the ordered triple $(s_1, s_2, 1)$ satisfies (2.1), and Young's inequality with general exponents $\mathbf{p}$ now follows directly from the special case with exponents $(s_1, s_2, 1)$.

If $\mathbf{f}$ is a $(1-\delta)$–near extremizer for the exponents $\mathbf{p}$, then

$$(1-\delta)\prod_{j=1}^3 \|f_j\|_{p_j} \le \langle f_1 * f_2, f_3\rangle \le \|f_1\|_{p_1}^{p_1/q}\|f_2\|_{p_2}^{p_2/q}\Big(\sum_{x,y} f_1(x)^{r_1} f_2(y)^{r_2} f_3(x+y)^{p_3}\Big)^{1/p_3}$$

and therefore

$$\sum_{x,y} f_1(x)^{r_1} f_2(y)^{r_2} f_3(x+y)^{p_3} \ge (1-\delta)^{p_3}\|f_1\|_{p_1}^{r_1}\|f_2\|_{p_2}^{r_2}\|f_3\|_{p_3}^{p_3}.$$

Equivalently, $\langle g_1 * g_2, g_3\rangle \ge (1-\delta)^{p_3}\prod_j \|g_j\|_{s_j}$. □

4. A consequence of strict uniform convexity

Extra information will later be derived from a study of near-extremizers in the special case when exactly one of the three indices p_j equals 1 — paradoxically, a case in which our main conclusions do not hold. In this section we exploit the strict uniform convexity of the unit ball of L^p for $p \in (1,\infty)$ to characterize near-extremizers $(f_j : j \in \mathbb{Z})$ of the triangle inequality $\|\sum_j f_j\|_p \le \sum_j \|f_j\|_p$.

Among the well-known inequalities of Clarkson [2] are the following two.

Lemma 4.1 (Clarkson's inequalities [2])**.** *Let $p \in (1,\infty)$ and set $q = p' = p/(p-1)$. Let $f, g \in \ell^p$. If $p \ge 2$ then*

$$\|f+g\|_p^p + \|f-g\|_p^p \le 2^{p-1}\|f\|_p^p + 2^{p-1}\|g\|_p^p.$$

If $p \le 2$ then

$$\|f+g\|_p^q + \|f-g\|_p^q \le 2(\|f\|_p^p + \|g\|_p^p)^{q-1}.$$

We will need the following direct consequence.

Corollary 4.2. *Let $p \in (1,\infty)$. Then there exist $C, \gamma \in \mathbb{R}^+$ such that for any $f, g \in L^p$ and any $\varepsilon \in [0,1]$, if*

$$\|f+g\|_p^p \ge (1-\varepsilon)\big(2^{p-1}\|f\|_p^p + 2^{p-1}\|g\|_p^p\big) \tag{4.1}$$

then

$$\|f-g\|_p^p \le C\varepsilon^\gamma\big(\|f\|_p^p + \|g\|_p^p\big). \tag{4.2}$$

Lemma 4.3. *For any $p \in (1,\infty)$, there exist $C, \rho \in \mathbb{R}^+$ with the following property. Let S be a nonempty countable index set. Let $\{f_i : i \in S\}$ be a set of L^p functions, at least one of which has positive norm. If $\sum_{i\in S}\|f_i\|_p < \infty$ and*

$$\|\sum_{i\in S} f_i\|_p \ge (1-\delta)\sum_{i\in S}\|f_i\|_p, \tag{4.3}$$

then the set S can be partitioned as a disjoint union $S = S' \cup S''$ so that the functions f_i and their sum $F = \sum_{i\in S} f_i$ satisfy

$$\sum_{i\in S''} \|f_i\|_p \le C\delta^\rho \sum_{i\in S} \|f_i\|_p \tag{4.4}$$

$$\Big\| f_i - \frac{\|f_i\|_p}{\|F\|_p} F \Big\|_p \le C\delta^\rho \|f_i\|_p \qquad \text{for all } i \in S'. \tag{4.5}$$

Proof. We may suppose without loss of generality that $\|F\|_p = 1$. There exists a bounded linear functional $\ell \in (L^p)^*$ which satisfies $\|\ell\|_{(L^p)^*} = 1$ and $\ell(F) = 1$. By discarding all those indices for which $\|f_i\|_p = 0$, we may assume that $\|f_i\|_p > 0$ for every $i \in S$. Set $F_i = f_i/\|f_i\|_p$. For each $i \in S$ decompose

$$F_i = s_i F + r_i$$

where $s_i = \ell(F_i) \in [-1, 1]$ and $r_i = F_i - \ell(F_i)F = F_i - s_i F$.

Now

$$\|r_i\|_p = \|F_i - s_i F\|_p \le C(1 - s_i)^\gamma \tag{4.6}$$

where $C, \gamma \in \mathbb{R}^+$ depend only on p. For

$$\|F_i + s_i F\|_p \ge \ell(F_i + s_i F) = 2s_i,$$

so

$$2^{p-1}\|F_i\|_p^p + 2^{p-1}\|s_i F\|_p^p - \|F_i + s_i F\|_p^p \le 2^{p-1} + 2^{p-1} - (2|s_i|)^p = 2^p(1 - |s_i|).$$

(4.6) thus follows from Corollary 4.2 with $f = F_i$ and $g = s_i F$.

Also

$$1 = \ell(F) = \sum_{i\in S} \ell(f_i) = \sum_{i\in S} s_i \|f_i\|_p$$

while

$$\sum_{i\in S} \|f_i\|_p \le (1-\delta)^{-1} \|\sum_{i\in S} f_i\|_p = (1-\delta)^{-1}\|F\|_p \le 1 + C\delta,$$

so

$$\sum_{i\in S} (1 - s_i)\|f_i\|_p \le C\delta.$$

Therefore

$$\sum_{i\in S} \|r_i\|_p^{1/\gamma} \|f_i\|_p \le C\delta,$$

where $C \in \mathbb{R}^+$ is a constant which depends only on p. Substituting $\|r_i\|_p = \|f_i\|_p^{-1}\|f_i - \ell(f_i)F\|_p$ gives

$$\sum_{i\in S} \Big(\frac{\|f_i - \ell(f_i)F\|_p}{\|f_i\|_p} \Big)^{1/\gamma} \|f_i\|_p \le C\delta.$$

Let $\eta > 0$ be a quantity to be chosen below, and define

$$S'' = \{ i \in S : \|f_i - \ell(f_i)F\|_p \ge \eta \|f_i\|_p \}$$

and of course $S' = S \setminus S''$. Then

$$\sum_{i \in S''} \eta^{1/\gamma} \|f_i\|_p \le C\delta,$$

so

$$\sum_{i \in S''} \|f_i\|_p \le C\delta\eta^{-1/\gamma}.$$

On the other hand, for every $i \in S'$ we have $\|f_i - \ell(f_i)F\|_p < \eta\|f_i\|_p$. Now

$$\|f_i - \|f_i\|_p F\|_p \le \|f_i - \ell(f_i)F\|_p + |\ell(f_i) - \|f_i\|_p|$$

The relation $F_i = s_i F + r_i$ can be rewritten as $f_i = \ell(f_i)F + r_i\|f_i\|_p$, so

$$\big|\|f_i\|_p - |\ell(f_i)|\big| = \big|\|f_i\|_p - \|\ell(f_i)F\|_p\big| \le \|r_i\|_p\|f_i\|_p = \|f_i - \ell(f_i)F\|_p.$$

If $i \in S'$ then, by definition of S', this is $\le \eta\|f_i\|_p$. Therefore $\|f_i - \|f_i\|_p F\|_p \le 2\eta\|f_i\|_p$.

Thus both conclusions (4.4),(4.5) of the lemma hold, with $\varepsilon(\delta) = \max(2\eta, C\delta\eta^{-1/\gamma})$. Choosing $\eta = \delta^{\gamma/(1+\gamma)}$ yields the required bounds. □

5. Conclusion of proof

All of the reasoning so far applies to any discrete group, with or without torsion. The remainder of the analysis relies on the following generalization of the Cauchy-Davenport inequality due to Kemperman [5]: For any two finite subsets A, B of any torsion-free group,

$$|A + B| \ge |A| + |B| - 1. \tag{5.1}$$

See also the alternative proof of Hamidoune [3].

The following quantity $\mathcal{N}$ quantifies the extent to which a function fails to be concentrated on any small set.

Definition 5.1. Let $p \in [1, \infty)$ and $\eta > 0$. For any nonzero $f \in L^p(G)$, $\mathcal{N}(f, \eta, p)$ denotes the largest integer N such that for all subsets $B \subset G$,

$$\|f\|^p_{L^p(G \setminus B)} < \eta\|f\|_p^p \ \Rightarrow\ |B| \ge N. \tag{5.2}$$

For any nonzero function f, $\mathcal{N}(f, \eta, p)$ is well-defined, and is ≥ 1. Moreover, $\mathcal{N}(f, \eta, p) = 1$ if and only if there exists z such that $\|f\|^p_{L^p(G\setminus\{z\})} \le \eta\|f\|_p^p$.

The main step in the proof of Theorem 1.1 is encapsulated in the next lemma.

Lemma 5.1. *Let* $\mathbf{p}$ *satisfy* (2.1) *and* (2.2). *For any* $\eta > 0$ *there exists* $\delta > 0$ *with the following property. Let* G *be any torsion-free discrete group. Let* $(f_1, f_2) \in (L^{p_1} \times L^{p_2})(G)$ *be any* $(1-\delta)$*–near extremizer. If* $\mathcal{N}(f_2, \eta, p_2) \ge 2$ *then*

$$\mathcal{N}(f_1, \eta, p_1) \ge 2\mathcal{N}(f_2, \eta, p_2). \tag{5.3}$$

Proof. Let C, ρ be the constants which appear in Lemma 4.3 for the exponent s_1, and suppose henceforth that δ is sufficiently small that $C\delta^\rho < \eta$. Set $N = \mathcal{N}(f_2, \eta, p_2)$.

For $u \in G$, let $\tau_u g(x) = g(x-u)$ denote the right translate of a function $g : G \to \mathbb{C}$. Since (f_1, f_2) is a $(1-\delta)$–near extremizing pair, Lemma 3.1 says that

$$\|\sum_{u \in G} f_2^{p_2}(u)\tau_u f_1^{r_1}\|_{s_1} \ge (1 - C_0\delta)\|f_1\|_{p_1}^{r_1}\|f_2\|_{p_2}^{p_2},$$

where s_j, r_j are the exponents which appear in that lemma and the constant C_0 depends on p_1, p_2 alone.

Apply Lemma 4.3 to the collection of functions $x \to \tau_u f_1^{r_1}(x) f_2^{p_2}(u)$ indexed by $u \in G$. Since $C\delta^\rho < \eta$, the lemma gives a function $\mathcal{F} \in L^{s_1}$ and a partition $G = S' \cup S''$ of G such that

$$\sum_{u \in S''} f_2^{p_2}(u) < \eta \|f_2\|_{p_2}^{p_2} \tag{5.4}$$

and

$$\|\tau_u f_1^{r_1} - \mathcal{F}\|_{s_1} < \eta \|f_1\|_{p_1}^{r_1} \text{ for all } u \in S'. \tag{5.5}$$

According to the definition (5.2) of $N = \mathcal{N}(f_2, \eta, p_2)$, (5.4) implies that $|S'| \geq N$. Inequality (5.5) implies that there exists a set T, which is a translate of S', such that

$$\|\tau_u f_1^{r_1} - f_1^{r_1}\|_{s_1} \leq 2\eta \|f_1\|_{p_1}^{r_1} \text{ for all } u \in T. \tag{5.6}$$

In particular, $|T| = |S'| \geq N$.

Let nT denote the set

$$nT = \{t_1 + t_2 + \cdots t_n : t_j \in T \text{ for all } j \in [1, n]\}$$

of all n-fold sums of elements of T. If $u \in nT$ then

$$\|\tau_u f_1^{r_1} - f_1^{r_1}\|_{s_1} \leq 2n\eta \|f_1\|_{p_1}^{r_1}.$$

Moreover, $|nT| \geq nN - n + 1$ by the Cauchy-Davenport inequality (5.1). Since $N \geq 2$, $|nT| \geq \frac{1}{2} nN$.

For any exponents $p \in [1, \infty)$ and $r \in [1, p]$ and any two nonnegative functions $g_i \in L^p$, $\|g_1 - g_2\|_{L^p}^p \leq C \|g_1^r - g_2^r\|_{L^{p/r}} \big(\|g_1\|_p + \|g_2\|_p\big)^{p-r}$. Therefore since $r_1 \cdot s_1 = p_1$,

$$\|\tau_u f_1 - f_1\|_{p_1} \lesssim n^{1/r_1} \eta^{1/r_1} \|f_1\|_{p_1} \text{ for all } u \in nT.$$

Recall the decomposition $f_1 = \sum_{k \in \mathbb{Z}} 2^k F_k$, the associated sets E_k, and the index $\kappa = \kappa_1$ of Lemma 2.1. Set $\tilde{E}_\kappa = E_{\kappa-1} \cup E_\kappa \cup E_{\kappa+1}$. For any $u \in G$,

$$\|\tau_u f_1 - f_1\|_{p_1} \geq \tfrac{1}{2} 2^\kappa \left|\tau_u E_\kappa \setminus \tilde{E}_\kappa\right|^{1/p_1},$$

where $A \setminus B$ denotes the intersection of a set A with the complement of a set B. One of the conclusions of Lemma 2.1 is that $2^\kappa |E_\kappa|^{1/p_1} \asymp \|f_1\|_{p_1}$ so long as δ does not exceed a certain positive constant which depends only on $\mathbf{p}$, not on η. Therefore for all sufficiently small η, for every $u \in nT$,

$$\left|\tau_u E_\kappa \setminus \tilde{E}_\kappa\right| \leq C(n\eta)^{p_1/r_1} |E_\kappa|. \tag{5.7}$$

The parameter n has not yet been specified. Choose it to satisfy $\frac{1}{4} \leq C(n\eta)^{p_1/r_1} \leq \frac{1}{2}$, where C is the constant in (5.7), assuming as we may that η is sufficiently small. Thus $n \asymp \eta^{-1}$.

For each $u \in nT$, $\left|\tau_u E_\kappa \cap \tilde{E}_\kappa\right| \geq \frac{1}{2} |E_\kappa|$, that is,

$$\int_{E_\kappa} \mathbf{1}_{\tilde{E}_\kappa}(\tau_{-u} x)\, dx \geq \tfrac{1}{2} |E_\kappa|.$$

Therefore

$$|E_\kappa|^{-1}\int_{E_\kappa}\sum_{u\in nT}\mathbf{1}_{\tilde E_\kappa}(\tau_{-u}x)\,dx \ge \tfrac12|nT|.$$

Consequently there exists $x\in E_\kappa$ which satisfies

$$\tau_{-u}(x)\in\tilde E_\kappa \text{ for at least } \tfrac12|nT| \text{ elements } u\in nT;$$

which is to say that $x+\tilde T\subset\tilde E_\kappa$ for some subset $\tilde T\subset nT$ of cardinality $\ge\frac12|nT|$. We conclude that

$$\big|\tilde E_\kappa\big|\ge\tfrac12|nT|\ge a\eta^{-1}N,$$

where a is a positive constant which depends only on $\mathbf{p}$.

Let $A\subset G$ be any subset of cardinality $\le\frac12 a\eta^{-1}N$. Then

$$\begin{aligned}\|f_1\|_{L^{p_1}(G\setminus A)} &\ge \|f_1\|_{L^{p_1}(\tilde E_\kappa\setminus A)}\\ &\ge 2^{\kappa-1}\big|\tilde E_\kappa\setminus A\big|^{1/p_1}\\ &\ge 2^{\kappa-1}2^{-1/p_1}\big|\tilde E_\kappa\big|^{1/p_1}\\ &\ge c_1\|f_1\|_{p_1}\end{aligned}$$

where $c_1>0$ is independent of f_1,η; the final inequality follows from (2.5). The contrapositive statement is that for any A,

$$\|f_1\|_{L^{p_1}(G\setminus A)}^{p_1}<c_1^{p_1}\|f_1\|_{p_1}^{p_1} \;\Rightarrow\; |A|>\tfrac a2\eta^{-1}N. \tag{5.8}$$

It is no loss of generality to assume that η is smaller than any specified constant, in particular, that $\eta<c_1^{p_1}$ and $\frac a2\eta^{-1}\ge 2$. Thus

$$\|f_1\|_{L^{p_1}(G\setminus A)}^{p_1}<\eta\|f_1\|_{p_1}^{p_1} \;\Rightarrow\; |A|\ge 2N=2\mathcal N(f_2,\eta,p_2),$$

concluding the proof of Lemma 5.1. □

Proof of Theorem 1.1. The proof of Lemma 5.1 applies equally well with the roles of f_1,f_2 reversed. Therefore if (f_1,f_2) is a $(1-\delta)$–near extremizer, if $\mathcal N(f_2,\eta,p_2)\ge 2$, and if δ is sufficiently small as a function of η and $\mathbf{p}$ alone, then

$$\mathcal N(f_2,\eta,p_2)\ge 2\mathcal N(f_1,\eta,p_1)\ge 4\mathcal N(f_2,\eta,p_2), \tag{5.9}$$

which is a contradiction.

Thus given η, if (f_1,f_2) is a $(1-\delta)$–near extremizer for sufficiently small δ, then $\mathcal N(f_2,\eta,p_2)=1$. The proof of Theorem 1.1 is complete. □

Because δ was constrained only by the bound $\delta\le C\eta^\rho$ in this argument, we have proved the following quantitative result, which establishes Theorem 1.2.

Proposition 5.2. *Let G be any torsion-free discrete group. Let $p_1,p_2\in(1,\infty)$ and suppose that $q^{-1}=p_1^{-1}+p_2^{-1}-1$ satisfies $q\in(1,\infty)$. There exist $c,\gamma\in\mathbb R^+$ such that if $\|f_1*f_2\|_q\ge(1-\delta)\|f_1\|_{p_1}\|f_2\|_{p_2}$, then*

$$\frac{\|f_i\|_\infty}{\|f_i\|_{p_i}}\ge 1-c\delta^\gamma$$

for both indices $i=1,2$.

6. The Hausdorff-Young inequality

Let $f \in \ell^p(\mathbb{Z}^d)$. In the case $p \le \frac{4}{3}$,

$$\|\widehat{f}\|_{p'}^2 = \|(\widehat{f})^2\|_{s'} = \|\widehat{f * f}\|_{s'} \le \|f * f\|_s \le \|f\|_p^2 \cdot \Lambda\Big(\frac{\|f\|_\infty}{\|f\|_p}\Big)^2 \tag{6.1}$$

where $s^{-1} = 2p^{-1} - 1$ and Λ is the same function which appears in Theorem 1.2.

Now consider the case $\frac{4}{3} < p < 2$. Let $\theta \in (0,1)$ satisfy $p^{-1} = \frac{\theta}{2} + \frac{3(1-\theta)}{4}$. Assume that $\|f\|_p = 1$ and write $f(x) = F(x)e^{i\varphi(x)}$ where φ is real-valued and $F(x) \ge 0$ for all x. Define $f_z(x) = F(x)^{L(z)}e^{i\varphi(x)}$ where $L : \mathbb{C} \to \mathbb{C}$ is the unique complex affine function which satisfies $L(\theta) = 1$, $\mathrm{Re}\,(L(z)) = p/2$ whenever $\mathrm{Re}\,(z) = 0$, and $\mathrm{Re}\,(L(z)) = 3p/4$ whenever $\mathrm{Re}\,(z) = 1$.

Then $\|\widehat{f_z}\|_2 = \|f_z\|_2 = 1$ whenever $\mathrm{Re}\,(z) = 0$ and, by inequality (6.1), $\|\widehat{f_z}\|_4 \le \|f_z\|_{4/3} \cdot \Lambda\Big(\frac{\|f_z\|_\infty}{\|f_z\|_{4/3}}\Big) = \Lambda\Big(\|f\|_\infty^{3p/4}\Big)$ whenever $\mathrm{Re}\,(z) = 1$. Therefore, by the three lines lemma,

$$\|\widehat{f}\|_{p'} = \|\widehat{f_\theta}\|_{p'} \le \Lambda\Big(\|f\|_\infty^{3p/4}\Big)^\theta$$

so for any $f \in \ell^p$,

$$\|\widehat{f}\|_{p'} \le \|f\|_p \cdot \widetilde{\Lambda}\Big(\frac{\|f\|_\infty}{\|f\|_p}\Big) \tag{6.2}$$

where $\widetilde{\Lambda}(t) = (\Lambda(t^{3p-4}))^\theta$ has the same character as Λ. □

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